Indian Statistical Institute
Back Paper Examination
Algebra II - BMath I

Max Marks: 100 Time: 180 minutes.

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

(1) Prove or disprove the following.
  (a) Let $T : M_{2\times 2}(\mathbb{R}) \longrightarrow V$ be a linear transformation where V is the vector space of 2×2 skew symmetric matrices. Then T cannot be one-one.
  (b) If V denote the vector space of all convergent real sequences, then V is finite dimensional.
  (c) If W and W' are subspaces of $\mathbb{R}^n$, then $\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W')$.
  (d) Let A and A' be matrices related by $PAQ = A'$ where P and Q are invertible. Then A and A' have the same rank.
  (e) Let A and A' denote the matrices of a symmetric bilinear form on a vector space V with respect to bases B and B' respectively. Then A and A' have the same eigenvalues. [5x3=15]

(2) If A, B are $n \times n$ matrices, show that $\det(AB) = \det(A) \cdot \det(B)$. [10]

(3) Let S and L be finite subsets of a finite dimensional vector space V. Assume that S spans V and that L is linearly independent. Then show that S contains at least as many elements as L. [10]

(4) Show that a vector space V over an infinite field cannot be written as the union of finitely many proper subspaces. [10]

(5) When do you say that a square matrix is nilpotent. Let A be a $n \times n$ nilpotent matrix. Show that $A^n = 0$. [15]

(6) Let $X, Y \in \mathbb{C}^n$ with $X \neq 0$. Then show that there exists a symmetric matrix B with $BX = Y$. [15]

(7) Prove that the only real matrix that is orthogonal, symmetric and positive definite is the identity. [10]

(8) Let V denote the vector space of all real polynomials of degree at most n. Check that for $f, g \in V$,
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$
defines a symmetric positive definite bilinear form on V. Find an orthonormal basis for V with respect to the above form when $n = 1, 2, 3$. [15]